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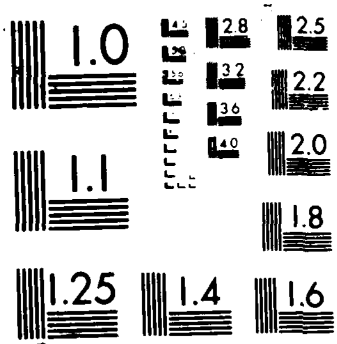
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NOTE ON BOUNDARY STABILIZATION OF WAVE EQUATIONS[1]

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Abstract. An energy decay rate is obtained for solutions of wave type equations in a bounded region in $\mathbb{R}^n$ whose boundary consists partly of a nontrapping reflecting surface and partly of an energy absorbing surface. Unlike most previous results on this problem, the results presented here are valid for regions having connected boundaries.

Key words. Wave equations, boundary stabilization, exponential stability.

Let Ω be a bounded, open, connected set in $\mathbb{R}^n$ $(n\geq 2)$ and Γ denote its boundary. Assume that Γ is piecewise smooth and consists of two parts, Γ_0 and Γ_1, with $\Gamma_1\neq\phi$ and relatively open in Γ, and Γ_0 either empty or having a non-empty interior. We set $\Sigma_0=\Gamma_0\times(0,\infty)$, $\Sigma_1=\Gamma_1\times(0,\infty)$. Let k be an $L^\infty(\Gamma_1)$ function satisfying $k(x)\geq 0$ almost everywhere on Γ_1. Consider the problem

(1) $$w''-\Delta w = 0 \quad \text{in } \Omega\times(0,\infty),$$

(2) $$\partial w/\partial \nu = -kw' \quad \text{on } \Sigma_1, \quad w = 0 \text{ on } \Sigma_0,$$

(3) $$w(0) = w^0, \quad w'(0) = w^1 \quad \text{in } \Omega$$

where $'=d/dt$ and ν is the unit normal of Γ pointing towards the exterior of Ω.

Associated with each solution of (1.1) is its total energy at time t:

$$E(t) = \frac{1}{2}\int_\Omega (w'^2 + |\nabla w|^2)dx.$$

A simple calculation shows that

$$E'(t) = -\int_{\Gamma_1} kw'^2 d\Gamma \leq 0,$$

hence E(t) is nonincreasing. The question of interest for us is the following: Under what conditions is it true that there is an exponential decay rate for E(t), i.e.,

(4) $$E(t) \leq Ce^{-\omega t}E(0), \qquad ,t\geq 0$$

for some positive ω.

The first person to establish (4) for solutions of (1)-(3) was G.

Chen [1], under the following assumptions: $k(x) \geq k_0 > 0$ on Γ_1, and there is a point $x_0 \in \mathbb{R}^n$ such that

(5) $$(x-x_0)\cdot\nu \leq 0, \quad x\in\Gamma_0,$$

(6) $$(x-x_0)\cdot\nu \geq \gamma > 0, \quad x\in\Gamma_1.$$

Chen slightly relaxed (5) and (6) in a later paper [2]. The most general result to date in terms of the assumed geometrical conditions on Γ appears in [5]. There it is proved that (4) is valid provided there exists a vector field $h(x)=[h_1(x),\cdots,h_n(x)]\in C^2(\bar{\Omega})$ such that

(7) $$h\cdot\nu \leq 0 \quad \text{on } \Gamma_0,$$

(8) $$h\cdot\nu \geq \gamma > 0 \quad \text{on } \Gamma_1,$$

(9) the matrix $(\partial h_i/\partial x_j + \partial h_j/\partial x_i)$ is positive definite on $\bar{\Omega}$.

This last result has subsequently been reproved by Lasiecka-Triggiani [7] and Triggiani [9] using methods different from those in [5]. In all of the papers cited, the estimate (4) was obtained from estimates on $\int_0^\infty E(t)dt$ by employing a result of Datko [3] (later extended by Pazy [8]). Thus in all cases the constants C and ω are <u>not</u> given explicitly in terms of problem data.

An important observation is that when Γ is smooth, the conditions (5) and (6) (resp., (7) and (8)) together force $\bar{\Gamma}_0\cap\bar{\Gamma}_1=\Phi$. Thus if $\Gamma_0\neq\phi$, the above results cannot apply to regions Ω having a connected boundary. However, in a recent paper [4], Kormornik and Zuazua succeeded in relaxing condition (6) of Chen to

(10) $$(x-x_0)\cdot\nu \geq 0 \quad \text{on } \Gamma_1$$

thus allowing for regions with smooth connected boundaries, but at the expense of replacing the boundary condition (2a) by

(11) $$\partial w/\partial\nu = -((x-x_0)\cdot\nu)w' \quad \text{on } \Sigma_1.$$

In addition, the proof in [4] gives *explicit* estimates of the constants C and ω in (4) in terms of the geometry of Ω, more specifically, in terms of the constants μ_0 and μ_1 which appear in (16), (17) below.

The purpose of this paper is to extend the result of [4] in two ways: first, by replacing the specific vector field $x-x_0$ in (5) and (10) by a general vector field h(x) satisfying (7), (9), and

(12) $$h\cdot\nu \geq 0 \quad \text{on } \Gamma_1,$$

and, second, by replacing the boundary condition (11) by

(13) $$\partial w/\partial\nu = -k^*(h\cdot\nu)w' \quad \text{on } \Sigma_1$$

where $k^*\in L^\infty(\Gamma_1)$ satisfies $k^*\geq k_0>0$ on Γ_1. Note that if $h\cdot\nu\geq\gamma>0$ on Γ_1, the boundary condition (2a) may be written as (13) with $k^*=k/(h\cdot\nu)$. Hence, in this situation, we recover (a sharpened form of) the main result of [5] (see Theorem below). Also, as in [4], we will obtain explicit estimates on the constants C and ω in (4) in terms on constants associated with the geometry of Ω, the gain k^* and the vector field h.

The formal statements of the two results to be proved are as follows.

THEOREM. *Let* w *be a regular solution to* (1), (2b) *and* (13). *Then there is a constant* ω (*which may be explicitly estimated*) *such that*

$$\int_0^\infty E(s)ds \leq (1/\omega)E(0),$$

$$\int_t^\infty E(s)ds \leq e^{-\omega t}\int_0^\infty E(s)ds, \quad t\geq o.$$

COROLLARY. *Under the hypotheses of the Theorem*,

$$E(t) \leq e\cdot e^{-\omega t}E(0), \quad t\geq 1/\omega.$$

Remark 1. If the initial data (3) satisfies $w^0\in H^1(\Omega)$, $w^1\in L^2(\Omega)$, w=0 on Γ_0, it is well known that (1)-(3) has a unique weak solution such that $(w,w')\in C([0,\infty);H^1(\Omega)\times L^2(\Omega))$, w=0 on Σ_0 in the sense of traces, and $k^{1/2}w'\in L^2(0,T;L^2(\Gamma_1))$, for every T>0. The proof of Theorem requires additional regularity of w, namely $(w,w')\in C([0,\infty);H^2(\Omega)\times H^1(\Omega))$. When $\bar{\Gamma}_0\cap\bar{\Gamma}_1\neq\phi$, this latter requirement may not be satisfied even for smooth data and boundary since singularities <u>may</u> develop at points on $\bar{\Gamma}_0\cap\bar{\Gamma}_1$. On the other hand, when $\bar{\Gamma}_0\cap\bar{\Gamma}_1=\phi$ the solution will always possess the necessary regularity if $w^0\in H^2(\Omega)$, $w^1\in H^1(\Omega)$, $w^0=0$ on Γ_0, $\partial w^0/\partial\nu+kw^1=0$ on Γ_1.

<u>Remark</u> 2. The Theorem and Corollary may be extended to generalized wave equations with time independent coefficients as in [5] but under the weaker condition (12) and also to linear elastodynamic systems (cf. p. 167 of [5] and also [6]). We omit details.

<u>Proof of Corollary</u>. Since $E(t)$ is nonincreasing, for every $\tau>0$

$$\tau E(t+\tau)\le\int_t^\infty E(s)ds\le(1/\omega)e^{-\omega t}E(0),$$

or

$$E(t+\tau)\le(e^{\omega\tau}/\omega\tau)e^{-\omega(t+\tau)}E(0),\quad \tau>0. \tag{14}$$

The first factor on the right has its minimum at $\tau=1/\omega$ and for this value of τ (14) becomes

$$E(t+1/\omega)\le e\cdot e^{-\omega(t+1/\omega)}E(0),\quad t\ge0.$$

<u>Proof of Theorem</u>. We assume that $\Gamma_0\neq\phi$. The argument may easily be modified to handle the opposite case as in [5] or [9].

Define the matrix $H=(\partial h_i/\partial x_j+\partial h_j/\partial x_i)$. By assumption we have

$$H\xi\cdot\xi\ge h_0|\xi|^2,\quad \xi\in\mathbb{R}^n,\ x\in\Omega,\ h_0>0. \tag{15}$$

Since multiplication of h by a positive constant leaves Γ_0 and Γ_1 invariant, we may (and do) assume that $h_0=1$ in (15).

Define constants μ_0 and μ_1 by

$$\int_{\Gamma_1}v^2dx\le\mu_0\int_\Omega|\nabla v|^2dx, \tag{16}$$

$$\int_\Omega v^2dx\le\mu_1\int_\Omega|\nabla v|^2dx \tag{17}$$

for all $v\in H^1(\Omega)$ such that $v=0$ on Γ_0 For $\epsilon>0$ and fixed, define

$$F_\epsilon(t) = E(t) + \epsilon\rho(t)$$

where

$$\rho(t) = 2(w',h\cdot\nabla w) + ((h_{j,j}-1)w,w').$$

We note that

$$|\rho(t)| \le C_0 E(t),$$

hence

(18) $$(1-\epsilon C_0)E(t) \le F_\epsilon(t) \le (1+\epsilon C_0)E(t)$$

where C_0 depends on h and μ_1. We will show that for ϵ sufficiently small,

(19) $$F_\epsilon'(t) \le -\epsilon E(t) + C\epsilon \int_\Omega w^2 dx$$

where C depends on h, μ_0 and μ_1.

One has

(20) $$\rho'(t) = 2(w'',h\cdot\nabla w) + 2(w',h\cdot\nabla w') + ((h_{j,j}-1)w',w') + ((h_{j,j}-1)w,w'').$$

From (1), (2) we have

(21) $$(w'',v) + (\nabla w,\nabla v) + b(w',v) - \int_{\Gamma_0} (\partial w/\partial \nu) v d\Gamma = 0$$

for every $v \in H^1(\Omega)$, where

$$b(w',v) = \int_{\Gamma_1} k^*(h\cdot\nu)w'v d\Gamma.$$

We use (21) to calculate $(w'',h\cdot\nabla w)$ and $((h_{j,j}-1)w,w'')$ in (20). One has

(22) $$(w'',h\cdot\nabla w) = -(\nabla w,\nabla(h\cdot\nabla w)) - b(w',h\cdot\nabla w) + \int_{\Gamma_0} (\partial w/\partial \nu)h\cdot\nabla w d\Gamma.$$

A direct calculation gives

(23) $$(\nabla w,\nabla(h\cdot\nabla w)) = \int_\Omega h_{i,j}w_i w_j dx - (1/2)\int_\Omega h_{j,j}|\nabla w|^2 dx + (1/2)\int_\Gamma h\cdot\nu|\nabla w|^2 d\Gamma.$$

Similarly,

(24) $$((h_{j,j}-1)w,w'') = -\int_\Omega (h_{j,j}-1)|\nabla w|^2 dx - \int_\Omega h_{j,ij} w w_i dx - b(w',(h_{j,j}-1)w).$$

We also have

$$(25)\qquad (w',h\cdot\nabla w') = (1/2)\int_{\Gamma_1}(h\cdot\nu)w'^2 d\Gamma - (1/2)\int_\Omega h_{j,j}w'^2 dx.$$

Use of (22) - (25) in (20) gives

$$(26)\qquad \rho'(t) = -2\int_\Omega h_{i,j}w_i w_j dx + \int_\Omega |\nabla w|^2 dx - \int_\Omega w'^2 dx - \int_\Omega h_{j,ij}ww_i dx - \int_\Gamma (h\cdot\nu)|\nabla w|^2 d\Gamma + 2\int_{\Gamma_0}(\partial w/\partial\nu)h\cdot\nabla w d\Gamma + \int_{\Gamma_1}(h\cdot\nu)w'^2 d\Gamma - 2b(w',h\cdot\nabla w) - b(w',(h_{j,j}-1)w).$$

The integrals over Γ_0, viz.

$$(27)\qquad 2\int_{\Gamma_0}(\partial w/\partial\nu)h\cdot\nabla w d\Gamma - \int_{\Gamma_0} h\cdot\nu|\nabla w|^2 d\Gamma = \int_{\Gamma_0} h\cdot\nu(\partial w/\partial\nu)^2 d\Gamma \le 0.$$

We also have the estimates

$$(28)\qquad |b(w',h\cdot\nabla w)| = |\int_{\Gamma_1} k^*(h\cdot\nu)w'(h\cdot\nabla w)d\Gamma| \le \int_{\Gamma_1} h\cdot\nu|\nabla w|^2 d\Gamma + C_1\int_{\Gamma_1}(h\cdot\nu)w'^2 d\Gamma,$$

$$(29)\qquad |b(w',(h_{j,j}-1)w)| \le C_2/(2\delta)\int_{\Gamma_1}(h\cdot\nu)w'^2 d\Gamma + (\delta/2)\int_\Omega |\nabla w|^2 dx,$$

$$(30)\qquad |\int_\Omega h_{j,ij}ww_i dx| \le C_3/(2\delta)\int_\Omega w^2 dx + (\delta/2)\mu_1\int|\nabla w|^2 dx$$

where C_1, C_2 depend on h and k^*, C_3 on h and where $\delta>0$ will be chosen below. Use of (27) - (30) and (15) (recall that $h_0=1$) in (26) yields

$$\rho'(t) \le -\int_\Omega (w'^2 + |\nabla w|^2)dx + (\delta/2)(\mu_0+\mu_1)\int_\Omega |\nabla w|^2 dx + (C_1+C_2/(2\delta) + 1)\int_{\Gamma_1}(h\cdot\nu)w'^2 d\Gamma + C_3/(2\delta)\int_\Omega w^2 dx.$$

Choosing $\delta = 1/(\mu_0+\mu_1)$ we obtain

$$(31)\qquad \rho'(t) \le -E(t) + C_4\int_{\Gamma_1}(h\cdot\nu)w'^2 dx + C_5\int_\Omega w^2 dx$$

where $C_4=C_1+C_2/(2\delta)+1$, $C_5=C_3/(2\delta)$. Since $k^*\ge k_0>0$ on Γ_1, we obtain from (31)

$$\begin{aligned} F_\epsilon'(t) &= E'(t) + \epsilon\rho'(t) \\ &= -\int_{\Gamma_1} k^*(h\cdot\nu)w'^2 d\Gamma + \epsilon\rho'(t) \\ &\le -\epsilon E(t) + \epsilon C_5\int_\Omega w^2 dx + \int_{\Gamma_1}(\epsilon C_4 - k_0)(h\cdot\nu)w'^2 d\Gamma \\ &\le -\epsilon E(t) + \epsilon C_5\int_\Omega w^2 dx \end{aligned}$$

provided $\epsilon C_4 \le k_0$. This establishes (19).

Let $\beta>0$ and consider

$$(32)\qquad \int_t^\infty e^{-\beta(s-t)}F_\epsilon'(s)ds = -F_\epsilon(t) + \beta\int_t^\infty e^{-\beta(s-t)}F_\epsilon(s)ds$$

$$\leq -\epsilon\int_t^\infty e^{-\beta(s-t)}E(s)ds + \epsilon C_5\int_t^\infty e^{-\beta(s-t)}|w(\cdot,s)|^2 ds.$$

From (18), $F_\epsilon(s)\geq 0$ provided $\epsilon C_0 \leq 1$. From Theorem 2 of [5], we have the estimate

$$(33)\qquad \int_t^\infty e^{-\beta(s-t)}|w(\cdot,s)|^2 ds \leq C_\eta^* E(t) + \eta\int_t^\infty e^{-\beta(t-s)}E(s)ds$$

where $\eta>0$ is arbitrary and C_η^* is a constant independent of β. Therefore (32), (33) imply

$$(34)\qquad \epsilon\int_t^\infty e^{-\beta(s-t)}E(s)ds \leq F_\epsilon(t) + \epsilon C_5[C_\eta^* E(t) + \eta\int_t^\infty e^{-\beta(s-t)}E(s)ds]$$

where $\epsilon=\min(1/C_0, k_0/C_4)$. Choosing $\eta=1/qC_5$ $(q>1)$ in (34) gives the estimate

$$(35)\qquad \frac{(q-1)\epsilon}{q}\int_t^\infty e^{-\beta(s-t)}E(s)ds \leq F_\epsilon(t) + \epsilon C_5 C_{1/q}^* E(t) \leq (1+\epsilon K_q)E(t)$$

where $K_q = C_0 + C_5 C_{1/q}^*$ does not depend on β. Define $\omega_q=(q-1)\epsilon/q(1+\epsilon K_q)$ and let $\beta\to 0$ in (35) to obtain

$$(36)\qquad \int_t^\infty E(s)ds \leq (1/\omega_q)E(t), \quad t\geq 0, \quad q>1.$$

The conclusions of the Theorem with $\omega=\omega_2=\epsilon/2(1+\epsilon K_2)$ (for example) follow easily from (36).

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